

The coloring game on matroids

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ABSTRACT. Given a loopless matroid M , one may define its chromatic number $\chi(M)$ as the minimum number of independent sets needed to partition the ground set of M . We consider a game-theoretic variant of this parameter defined as follows. Suppose that Alice and Bob alternately color the elements of M using a fixed set of colors. The only restriction is that, at each stage of the play, the elements of the same color must form an independent set. The game stops when either the whole matroid has been colored, or there is no admissible move. Alice wins in the former case, while Bob wins in the later. The minimum size of the set of colors for which Alice has a winning strategy is called the *game chromatic number* of M , denoted by $\chi_g(M)$.

We prove that $\chi_g(M) \leq 2\chi(M)$ for every matroid M . This improves the result of Bartnicki, Grytczuk and Kierstead [2], who proved that $\chi_g(M) \leq 3\chi(M)$ for graphic matroids. We also improve slightly the lower bound of [2] by giving a different class of matroids satisfying $\chi(M) = k$ and $\chi_g(M) = 2k - 1$, for every $k \geq 3$.

Our main result holds in a more general setting of *list colorings*. As an accompanying result we give a necessary and sufficient condition for a matroid to be list colorable from any lists of fixed sizes. This extends a theorem of Seymour [13] asserting that every matroid is colorable from arbitrary lists of size $\chi(M)$.

1. Introduction

Let M be a loopless matroid on a ground set E (the reader is referred to [9] for background of matroid theory). The *chromatic number* of M , denoted by $\chi(M)$, is the minimum number of independent sets of M needed to partition the set E . In case of a graphic matroid $M = M(G)$, the number $\chi(M)$ is known as the *edge arboricity* of the underlying graph G .

In this paper we study a game-theoretic variant of $\chi(M)$ defined as follows (for graphs it was introduced independently by Brams ,cf. [8], and Bodlaender [4]). Two players, Alice and Bob, alternately assign colors from a fixed set C to the elements of E . The only rule that both players have to obey is that at any moment of the play, all elements in the same color must form an independent set. Alice wins if the whole matroid has been colored successfully, while Bob wins if they arrive to a partial coloring that cannot be further extended (what happens when trying to color any uncolored element with any possible color results in a monochromatic

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cycle of M). Let $\chi_g(M)$ denote the minimum size of the set of colors C guaranteeing a win for Alice. This parameter is a natural, matroidal version of a well studied graph parameter—the game chromatic number $\chi_g(G)$ (see [1, 3, 6, 10, 15]).

The first step in studying $\chi_g(M)$ was made by Bartnicki, Grytczuk, and Kierstead [2], who proved that $\chi_g(M) \leq 3\chi(M)$ for graphic matroids M . We improve and extend this result by showing that $\chi_g(M) \leq 2\chi(M)$ holds for every matroid M . This gives a nearly tight bound, since there are graphic matroids with $\chi(M) = k$ and $\chi_g(M) \geq 2k - 2$, for every $k \geq 1$ [2]. We improve slightly this result by providing a different class of matroids satisfying $\chi(M) = k$ and $\chi_g(M) = 2k - 1$ for every $k \geq 3$.

The method we use is much different than that of [2]. We were inspired by an elegant result of Seymour, who proved that the choice number of any matroid is the same as its chromatic number. We extend this theorem by giving a necessary and sufficient condition for a matroid M to be colorable from arbitrary lists of fixed sizes. As an application of this result we obtain a similar upper bound for the list version of parameter $\chi_g(M)$ and a simple proof of some basis exchange properties.

2. A strategy for Alice

Let M be a loopless matroid on a ground set E , with a rank function r . A collection of (not necessarily different) subsets $V_1, \dots, V_d \subseteq E$ is said to be a k -covering of E if for each element $e \in E$ there are exactly k members of the collection containing e . As defined in the introduction the *chromatic number* of M , denoted by $\chi(M)$, is the minimum number of independent sets of M needed to 1-cover the set E . The *fractional chromatic number* of M , denoted by $\chi_f(M)$, is the infimum of fractions $\frac{b}{a}$, such that M has an a -covering with b independent sets. Extending a theorem of Nash-Williams [12] for graph arboricity, Edmonds [7] proved the following formula for the chromatic number of general matroids (and as it was observed his proof shows also a formula for fractional chromatic number):

$$\chi_f(M) = \max_{A \subseteq E} \frac{|A|}{r(A)}, \quad \chi(M) = \lceil \chi_f(M) \rceil.$$

We shall need a more general version of the matroid coloring game. Let M_1, \dots, M_d be a collection of matroids on the same ground set E . The set of colors is restricted to $\{1, \dots, d\}$. As before, the players alternately color the elements of E , but now the set of elements color by i must be independent in matroid M_i , for all $i \in \{1, \dots, d\}$. The other difference is that each element receives not only one color, but at most k colors. Alice wins if the whole matroid has been k -colored successfully, each element received k colors, while Bob wins if they arrive to a coloring that cannot be further extended and some elements have less than k colors. We will call this game a k -coloring game on M_1, \dots, M_d . The initial game on M coincides with the 1-coloring game on $M = M_1 = \dots = M_d$.

We can consider a fractional game chromatic number of a matroid M . For a given integers a and b consider a slight modification of the usual game. Alice and Bob, alternately assign colors from a fixed set C of b colors to the elements of E . The rule that both players have to obey stays unchanged, at any moment of the play, all elements in the same color must form an independent set. The only difference is that each element has to receive a colors. This game on M coincides with the a -coloring game on $M = M_1 = \dots = M_b$. The infimum of fractions $\frac{b}{a}$ for

which Alice has a winning strategy is called the *fractional game chromatic number* of M , which we denote by $\chi_{f,g}(M)$. Clearly, $\chi_{f,g}(M) \leq \chi_g(M)$ for every M .

THEOREM 1. *Let M_1, \dots, M_d be matroids on the same ground set E . If E has a $2k$ -covering by sets V_1, \dots, V_d , with V_i independent in M_i , then Alice has a winning strategy in the matroid k -coloring game on M_1, \dots, M_d . In particular, every matroid M satisfies $\chi_g(M) \leq 2\chi(M)$ and $\chi_{f,g}(M) \leq 2\chi_f(M)$.*

PROOF. Let us fix a $2k$ -covering by independent sets V_1, \dots, V_d . Alice will be coloring elements from V_i always by i , while Bob may not. At any point of the game let U_i denote the set of elements colored by i . Sets V_i, U_i will change during the game (former ones will decrease, while later ones will increase), however we will keep the invariant that after Alice's move (or at the beginning) the following three conditions are satisfied:

- (1) for each i sets V_i and U_i are disjoint,
- (2) for each i the set $U_i \cup V_i$ is independent in M_i ,
- (3) for each $e \in E$ the number $w(e)$ of sets V_i containing e and the number $c(e)$ of colors of e satisfy $w(e) + 2c(e) = 2k$.

At the beginning of the game they are clearly satisfied. To prove that Alice can keep the above conditions let us assume that Bob has colored an element x by color j . There are two possibilities.

If $U_j \cup \{x\} \cup V_j$ is an independent set in M_j , and if for some y there is $c(y) < k$ then by condition (3) we have $y \in V_l$. Alice can color y by l . If necessary we simply narrow sets V_i , to keep conditions (1) – (3) hold. Otherwise if $c(y) \geq k$ for all $y \in E$ Alice has won.

When $U_j \cup \{x\} \cup V_j$ is dependent in M_j , then by using the augmentation property we extend the independent set $U_j \cup \{x\}$ from the set $U_j \cup V_j$ in M_j . The extension equals to $U_j \cup \{x\} \cup V_j \setminus \{y\}$ for some $y \in V_j$. Since $w(y) + 2c(y) = 2k$ and $w(y) \geq 1$, we know that $w(y) \geq 2$, so $y \in V_l$ for some $l \neq j$. Now Alice has an admissible move, and her strategy is to color y by l . Let us observe that after this pair of moves, and possibly narrowing some sets V_i , the conditions (1) – (3) still hold. Indeed $U'_j \cup V'_j = U_j \cup \{x\} \cup V_j \setminus \{y\}$ is independent in M_j , and for y the value of $c(y)$ increased by 1 (by adding color l) and $w(e)$ decreased by 2 (by excluding y from V_j and from V_l).

To end the proof of the first part it is enough to observe that condition (3) implies that if Alice has not won yet then there is an admissible move for Bob.

To get the second assertion, let $\chi(M) = k$. So M has a partition into independent sets V_1, \dots, V_k . We may take $M_1 = \dots = M_{2k} = M$ and repeat each set V_i to get a 2 -covering of E . By the first assertion of the theorem we infer that Alice has a winning strategy and therefore $\chi_g(M) \leq 2k$.

To get the last assertion, suppose that M has a a -covering by b independent sets V_1, \dots, V_b (that is $\chi_f(M) \leq \frac{b}{a}$). We may take $M_1 = \dots = M_{2b} = M$ and repeat each set V_i to get a $2a$ -covering of E . By the first assertion of the theorem we get that Alice has a winning strategy and therefore $\chi_{f,g}(M) \leq 2\frac{b}{a}$. \square

Motivated by the result of Edmonds we ask the following:

QUESTION 1. *What is the relation between $\chi_g(M)$ and $\chi_{f,g}(M)$? Can the difference between them be arbitrary large?*

As usually in this kind of games the following question seems to be natural and non trivial (for graph coloring game it was asked by Zhu [15]).

QUESTION 2. *Suppose Alice has a winning strategy on a matroid M with k colors. Does she also have it with $l > k$ colors?*

3. A generalization of Seymour's Theorem

Let M be a matroid on a ground set E . By a *list assignment* we mean any function $L : E \rightarrow \mathcal{P}(\mathbb{N})$. A matroid M is said to be *colorable from lists L* if there is a proper coloring (elements of the same color form an independent set) $c : E \rightarrow \mathbb{N}$ of M such that $c(e) \in L(e)$ for every $e \in E$. The *choice number* $\text{ch}(M)$ is defined as the least number k such that M is colorable from any lists of sizes at least k . In [13] Seymour proved the following result as a simple application of the matroid union theorem.

THEOREM 2. (Seymour [13]) *Every loopless matroid M satisfies $\text{ch}(M) = \chi(M)$.*

In the sequel we shall generalize this theorem. To this end we will need a weighted version of the matroid union theorem. Let $W : E \rightarrow \mathbb{N}$ be a weight assignment for elements of E . A collection of (not necessarily different) subsets $V_1, \dots, V_d \subseteq E$ is said to be a *W -covering* of E if for each element $e \in E$ there are exactly $W(e)$ members of the collection containing e .

LEMMA 1. *Let M_1, \dots, M_d be matroids on the same ground set E with rank functions r_1, \dots, r_d . The following conditions are equivalent:*

- (1) *There exists a W -covering of E by sets V_1, \dots, V_d such that V_i is independent in M_i .*
- (2) *For each $A \subset E$ holds $r_1(A) + \dots + r_d(A) \geq \sum_{e \in A} W(e)$.*

PROOF. Implication (1) \Rightarrow (2) is easy. We focus on the opposite and prove it by double induction, on size of E and on the sum $\sum_i \sum_{A \subset E} r_i(A)$. Consider a function

$$f(A) = r_1(A) + \dots + r_d(A) - \sum_{e \in A} W(e).$$

Notice that it is nonnegative for $A \subset E$.

If for some proper nonempty subset $A \subset E$ we have $f(A) = 0$, then consider matroids M_1, \dots, M_d restricted to A . Since the assumption of the lemma is satisfied, so from the inductive assumption there is a $W|_A$ -covering U_1, \dots, U_d . Consider also matroids M'_1, \dots, M'_d on $E \setminus A$ obtained by contracting set A , that is $M'_i = M_i/A$ and $r'_i(X) = r_i(X \cup A) - r_i(A)$. They also satisfy the assumptions of the lemma, so from the inductive assumption there is a $W|_{E \setminus A}$ -covering U'_1, \dots, U'_d . Now the sums $V_i = U_i \cup U'_i$ form a W -covering, where V_i is independent in M_i .

Otherwise for any proper nonempty subset $A \subset E$ we have $f(A) > 0$. If there exists i and $e \in E$ such that $r_i(E) = r_i(E \setminus \{e\})$ and $r_i(e) = 1$, then we consider matroids $M_1, \dots, M'_i, \dots, M_d$ where $r'_i(X) = r_i(X \setminus \{e\})$. Assumptions of the lemma are still satisfied, so from the inductive assumption there is a W -covering. Obviously it is also a W -covering for matroids M_1, \dots, M_d .

Otherwise for each matroid M_i there is a basis $B_i \subset E$ such that $r_i(E \setminus B_i) = 0$. In this case sets B_1, \dots, B_d form at least a W -covering. Because for any $e \in E$ there is $r_1(e) + \dots + r_d(e) = |\{i : e \in B_i\}| \geq W(e)$. Clearly we can choose exactly a W -covering. \square

Let $L : E \rightarrow \mathcal{P}(\mathbb{N})$ be a list assignment. By the *size* of L we mean the function $\ell : E \rightarrow \mathbb{N}$ such that $|L(e)| = \ell(e)$ for each $e \in E$. For a weight assignment W and list assignment L we say that M is *W -colorable from lists L* , if it is possible to choose for each $e \in E$ a set of $W(e)$ colors from its list $L(e)$, such that elements with the same color form an independent set in M . In other words each list $L(e)$ contains a subset $L'(e)$ of size $W(e)$ such that choosing any color from $L'(e)$ for e results in a proper coloring of a matroid M .

THEOREM 3. *Let M be a matroid on a set E , and let $\ell : E \rightarrow \mathbb{N}$ and $W : E \rightarrow \mathbb{N}$ be two fixed functions. Then the following conditions are equivalent:*

- (1) *For every list assignment L of size ℓ , matroid M is W -colorable from L .*
- (2) *Matroid M is W -colorable from lists $\mathbf{L}(e) = \{1, \dots, \ell(e)\}$.*
- (3) *For each $A \subseteq E$ holds $\sum_{i \in \mathbb{N}_+} r(\{e \in A : \ell(e) \geq i\}) \geq \sum_{e \in A} W(e)$.*

PROOF. Let L be a fixed list assignment of size ℓ , let us denote $Q_i = \{e \in E : i \in L(e)\}$, and $d = \max \bigcup_{e \in E} L(e)$ (and respectively \mathbf{Q}_i and \mathbf{d} for lists \mathbf{L}). Consider matroids M_1, \dots, M_d , such that $M_i = M|_{Q_i}$. Observe that matroid M is properly W -colorable from lists L if and only if its ground set E can be W -covered by sets V_1, \dots, V_d , such that V_i is independent in M_i . Namely two expressions to cover and to color have the same meaning. From Lemma 1 we get that this condition is equivalent to the fact, that for each $A \subset E$ there is an inequality $r(A \cap Q_1) + \dots + r(A \cap Q_d) \geq \sum_{e \in A} W(e)$.

Now equivalence between conditions (2) and (3) follows from the fact that $\{e \in A : \ell(e) \geq i\} = A \cap \mathbf{Q}_i$.

Implication (1) \Rightarrow (2) is obvious. To prove (2) \Rightarrow (1) it is enough to show that for each $A \subset E$ there is an inequality

$$r(A \cap Q_1) + \dots + r(A \cap Q_d) \geq r(A \cap \mathbf{Q}_1) + \dots + r(A \cap \mathbf{Q}_d).$$

We know that $\bigcup_i Q_i = \bigcup_i \mathbf{Q}_i = \bigcup_{e \in E} W(e)\{e\}$ as multisets. Observe that when we want to prove the inequality, then whenever there are Q_i and Q_j , one not contained in the other, we can replace them by sets $Q_i \cup Q_j$, and $Q_i \cap Q_j$. This is because the following conditions are satisfied:

- $r(A \cap Q_i) + r(A \cap Q_j) \geq r(A \cap (Q_i \cup Q_j)) + r(A \cap (Q_i \cap Q_j))$, which follows from the submodularity
- $Q_i + Q_j = (Q_i \cup Q_j) + (Q_i \cap Q_j)$ as multisets
- $|Q_i|^2 + |Q_j|^2 < |Q_i \cup Q_j|^2 + |Q_i \cap Q_j|^2$.

The last parameter grows, and it is bounded (because the number of non empty sets, and size of each set are bounded). Hence after finite number of steps replacement procedure ends. Then Q_i are linearly ordered by inclusion. Let us reorder them as $Q_1 \supset Q_2 \supset \dots$. It is easy to see that $Q_1 = \mathbf{Q}_1$, $Q_2 = \mathbf{Q}_2$, \dots , so the inequality $r(A \cap Q_1) + \dots + r(A \cap Q_d) \geq r(A \cap \mathbf{Q}_1) + \dots + r(A \cap \mathbf{Q}_d)$ holds. This proves the assertion of the theorem. \square

The theorem of Seymour follows from the above result by taking a constant size function $\ell \equiv k$ and a weight function $W \equiv 1$. It also implies the following stronger statement.

COROLLARY 1. *Let I_1, \dots, I_k be a partition of the ground set of a matroid M into independent sets, and let $\ell(e) = i$ if $e \in I_i$. Then matroid M is properly colorable from any lists of size ℓ .*

Suitable choice of lists and their sizes leads to simple proofs of some basis exchange properties (for other proofs see [11, 14]).

COROLLARY 2. (multiple basis exchange) *Let B_1 and B_2 be bases of a matroid M . Then for every $A_1 \subset B_1$ there exists $A_2 \subset B_2$, such that $(B_1 \setminus A_1) \cup A_2$ and $(B_2 \setminus A_2) \cup A_1$ are both bases.*

PROOF. Observe that we can restrict to the case when $B_1 \cap B_2 = \emptyset$. Indeed otherwise consider matroid $M/(B_1 \cap B_2)$, its disjoint bases $B_1 \setminus B_2, B_2 \setminus B_1$ and $A_1 \setminus B_2 \subset B_1 \setminus B_2$. For them there exists appropriate A_2 , now $A_2 \cup (A_1 \cap B_2)$ solves the original problem.

If bases B_1 and B_2 are disjoint then restrict matroid M to their sum. Let L assigns lists $\{1\}$ to elements of A_1 , lists $\{2\}$ to elements of $B_1 \setminus A_1$ and lists $\{1, 2\}$ to elements of B_2 . Observe that for $W \equiv 1$ the condition (3) of Theorem 3 is satisfied, so we get that there is a 1-coloring from lists L . Denote by C_1 elements colored by 1, and by C_2 those colored by 2. Now $A_2 = C_2 \cap B_2$ is good, since $(B_1 \setminus A_1) \cup A_2 = C_2$ and sets $(B_2 \setminus A_2) \cup A_1 = C_1$ are independent. \square

COROLLARY 3. *Let A and B be basis of a matroid M . Then for every partition $A_1 \sqcup \dots \sqcup B_k = B$ there exists a partition $A_1 \sqcup \dots \sqcup A_k = A$, such that $(B \setminus B_i) \cup A_i$ are all bases for $1 \leq i \leq k$.*

PROOF. Analogously to the proof of Corollary 2 we can assume that bases A and B are disjoint (on their intersection sets A_i and B_i should be equal).

If bases A and B are disjoint then restrict matroid M to their sum. Let L assigns lists $\{1, \dots, k\} \setminus \{i\}$ to elements of B_i , and lists $\{1, \dots, k\}$ to elements of A . Let $W \equiv k - 1$ on B and $W \equiv 1$ on A . Observe that the condition (3) of Theorem 3 is satisfied, so we get that there is a W -coloring from lists L . Denote by C_i elements colored by i . Now $A_i = C_i \cap A$ is a good partition, since the sets $(B \setminus B_i) \cup A_i = C_i$ are independent. \square

COROLLARY 4. *Let A and B be basis of a matroid M . Then for every partition $B_1 \sqcup \dots \sqcup B_k = B$ there exists a partition $A_1 \sqcup \dots \sqcup A_k = A$, such that $(A \setminus A_i) \cup B_i$ are all bases for $1 \leq i \leq k$.*

PROOF. Analogously to the proofs of previous corollaries we can assume that bases A and B are disjoint. Then we restrict matroid M to $A \cup B$. Let L assigns lists $\{i\}$ to elements of B_i , and lists $\{1, \dots, k\}$ to elements of A . Let $W \equiv 1$ on B and $W \equiv k - 1$ on A . Observe that the condition (3) of Theorem 3 holds, so we get that there is a W -coloring from lists L . Denote by C_i elements colored by i . Now $A_i = A \setminus C_i$ is a good partition, since the sets $(A \setminus A_i) \cup B_i = C_i$ are independent. \square

4. A list version

We can consider a game list chromatic number of a matroid M . For a given k -element list assignment of the ground set E consider a slight modification of the usual game. Alice and Bob alternately color uncolored elements of E in such a way that each vertex receives a color from its list and it is a proper coloring. Alice wins if the whole matroid has been colored, while Bob wins if a partial coloring cannot be properly extended. The minimum number k for which Alice has a winning strategy for every k -element list assignment is called a *game list chromatic number* of M , which we denote by $\text{ch}_g(M)$. Clearly, $\text{ch}_g(M) \geq \chi_g(M)$ for every M .

COROLLARY 5. *Every matroid M satisfies $\text{ch}_g(M) \leq 2\chi(M)$.*

PROOF. Let $\chi(M) = k$, and $L : E \rightarrow L_e$ be a $2k$ -element list assignment. Denote also $Q_i = \{e : i \in L_e\}$ and $M_i = M|_{Q_i}$. From the implication (2) \Rightarrow (1) from Theorem 3 for $\ell \equiv k$, and $W \equiv 2$ there is a 2-covering V_1, \dots, V_d of E consisting of sets independent in the matroids M_i . So by Theorem 1, Alice has a winning strategy and as a consequence, $\text{ch}_g(M) \leq 2k$. \square

We can even define a fractional game list chromatic number of a matroid M . For a given integers a, b and a given b -element list assignment of the ground set E consider a the following game. Alice and Bob, alternately assign colors in such a way that each vertex receives a color from its list. The rule that both players have to obey stays unchanged, at any moment of the play, all elements in the same color must form an independent set. The only difference is that each element has to receive a colors. The infimum of fractions $\frac{b}{a}$ for which Alice has a winning strategy is called a *fractional game list chromatic number* of M , which we denote by $\text{ch}_{f,g}(M)$. Clearly, $\text{ch}_{f,g}(M) \leq \text{ch}_g(M)$ for every matroid M .

COROLLARY 6. *Every matroid M satisfies $\text{ch}_{f,g}(M) \leq 2\chi_f(M)$.*

PROOF. Suppose that M has a a -covering by b independent sets V_1, \dots, V_b (that is $\chi_f(M) \leq \frac{b}{a}$). Let $L : E \rightarrow L_e$ be a $2b$ -element list assignment, denote also $Q_i = \{e : i \in L_e\}$ and $M_i = M|_{Q_i}$. From the implication (2) \Rightarrow (1) from Theorem 3 for $\ell \equiv b$, and $W \equiv 2a$ there is a $2a$ -covering V_1, \dots, V_d of E consisting of sets independent in the matroids M_i . So by Theorem 1, Alice has a winning strategy and as a consequence, $\text{ch}_g(M) \leq 2\frac{b}{a}$. \square

Let $L : E \rightarrow \mathcal{P}(\mathbb{N})$ be a list assignment on the ground set of matroid M . We can even consider usual game with a restriction for both players that each vertex e can be colored only with a color from its list $L(e)$. So it is a variant of the above game where the lists are fixed, but not necessary of equal size. We denote such game by $\mathcal{G}(M)_L$.

COROLLARY 7. *If a matroid M , with size of lists assignment ℓ , is 2-colorable from lists $\mathbf{L}(e) = \{1, \dots, \ell(e)\}$, then for any list assignment L of size ℓ Alice has a winning strategy in the $\mathcal{G}(M)_L$ game.*

PROOF. Denote $Q_i = \{e : i \in L_e\}$. From the implication (2) \Rightarrow (1) from Theorem 3 for ℓ , and $W \equiv 2$ there is a 2-covering V_1, \dots, V_d of sets independent in the matroids $M_i = M|_{Q_i}$. So by Theorem 1 Alice has a winning strategy in the $\mathcal{G}(M)_L$ game. \square

We give a possible application of the above corollary.

COROLLARY 8. *Let G be a graph, and let ℓ be a function defined by*

$$\ell : E(G) \ni e = \{u, v\} \rightarrow \max\{\deg(u), \deg(v)\} + 1 \in \mathbb{N}.$$

Then for any list assignment L of size ℓ Alice has a winning strategy in the $\mathcal{G}(M(G))_L$ game (game in which both players color edges of the graph G from their lists avoiding monochromatic cycles).

PROOF. Due to Corollary 7 it is enough to show that $M(G)$ is 2-colorable from lists $\mathbf{L}(e) = \{1, \dots, \ell(e)\}$. From Theorem 3 it is equivalent that for each $A \subset E(G)$ holds $\sum_{i \in \mathbb{N}_+} r(\{e \in A : \ell(e) \geq i\}) \geq 2|A|$. We are going to prove these inequalities.

When we consider $G(A)$ a subgraph of G defined by all edges from A , then in the inequality for $E(G(A)) = A$, the right hand side is the same, while the left hand side may decrease. Hence we see, that it is enough to show the inequality for $A = E(G)$. It is worth to point out that for cliques there is an equality.

It is enough to show the inequality for each connected component, so let us assume that G is connected and $|V(G)| = n$. For $i = 1$ we have

$$r(\{e \in E(G) : l(e) \geq 1\}) = r(E(G)) = n - 1.$$

Observe that $\max_{v \in V(G)} \deg(v) \leq n - 1$, so

$$\sum_{i \in \mathbb{N}_+} r(\{e \in E(G) : \ell(e) \geq i\}) = \sum_{1 \leq i \leq n-1} (r(\{e \in E(G) : \ell(e) - 1 \geq i\}) + 1).$$

On the other side

$$2|E(G)| = \sum_{v \in V(G)} \deg(v) = \sum_{1 \leq i \leq n-1} |\{v \in V(G) : \deg(v) \geq i\}|.$$

We show that for each $1 \leq i \leq n - 1$ the following inequality holds:

$$r(\{e \in E(G) : \ell(e) - 1 \geq i\}) + 1 \geq |\{v \in V(G) : \deg(v) \geq i\}|.$$

Let G_i a subgraph of G defined by all edges with $\ell(e) - 1 \geq i$. Directly from the definition we get that G_i contains all vertices v with $\deg(v) \geq i$. If H is a connected component of G_i not containing all vertices of G , then H contains a vertex w with $\deg(w) < i$. For each such component

$$r(\{e \in H\}) = |V(H)| - 1 \geq |\{v \in V(H) : \deg(v) \geq i\}|.$$

Otherwise if G_i is connected, and contains all vertices of G , then

$$r(\{e \in E(G) : \ell(e) - 1 \geq i\}) + 1 = n \geq |\{v \in V(G) : \deg(v) \geq i\}|.$$

This completes the proof. \square

Inspired by Seymour's Theorem we ask the following:

QUESTION 3. *Is it true that for any matroid M the game list chromatic number equals to the game chromatic number ($\text{ch}_g(M) = \chi_g(M)$)?*

5. A strategy for Bob

We present a family of matroids M_k for $k > 2$, such that $\chi(M_k) = k$ and $\chi_g(M_k) \geq 2k - 1$. This improves the lower bound from the paper [2] of Bartnicki, Grytczuk and Kierstead. They gave an example of graphical matroids H_k with $\chi(H_k) = k$ and $\chi_g(H_k) \geq 2k - 2$ for each k . We believe that their example also satisfies $\chi_g(H_k) \geq 2k - 1$ for $k > 2$, however the proof would be much longer and more technical than in our case.

Let $k > 2$, and M_k be a transversal matroid on the set $E_k = C \cup D_1 \cup \dots \cup D_{3k(2k-1)}$, where $C, D_1, \dots, D_{3k(2k-1)}$ are disjoint sets, $C = \{c_{1,1}, \dots, c_{k,2k-1}\}$ has $k(2k-1)$ elements, and each $D_i = \{d_{1,i}, \dots, d_{k,i}\}$ has k elements. Subsets of E_k on which M_k consists of transversals are $D_1, \dots, D_{3k(2k-1)}$ and $(2k-1)$ copies of E .

Observe that $\chi(M_k) = k$, and also $\chi_f(M_k) = k$. Clearly we can take k independent sets $V_i = \{c_{i,1}, \dots, c_{i,2k-1}, d_{i,1}, \dots, d_{i,3k(2k-1)}\}$. Also observe that the rank of $C \cup D_i$ equals to $2k$, so if there are t elements from D_i colored by i , then there are at most $2k - t$ elements from C colored by i . This suggests that Bob should try to color D_i 's with one color.

THEOREM 4. *For $k > 2$, every matroid M_k satisfies $\chi_g(M_k) \geq 2k - 1$.*

PROOF. Suppose Alice has a winning strategy using $h \leq 2k - 2$ colors from the set $\{1, \dots, h\}$. We show a strategy for Bob, on which she fails. Assume first that Alice colors only elements from C , which is the main case to understand. Bob wants to keep the following invariant. For any $i \leq h$, always after his move, the number d_i of elements colored by i in D_i is greater or equal to the number c_i of elements colored with i in C . It is easy to see that he can do it. It can be proven by induction. At the beginning the condition is satisfied. From the inductive assumption after his move it is also satisfied, so after Alice's move (she colored some $c \in C$ with j) it can be wrong only by one for j . So he colors an element of D_j with j , and the condition is still satisfied. He can do it because if $c_j = d_j + 1$, and $c_j + d_j \leq 2k$, then also $c_j + (d_j + 1) \leq 2k$ (so elements colored with i are independent), and $d_j + 1 \leq k$ (there was uncolored element in D_i). In particular this was not the end of the game. If after her move the invariant is satisfied, and there exists $i \leq h$, such that $d_i < k$, then he colors an element of D_i with i . He can do it since $c_i \leq d_i$, so $c_i + d_i < 2k$ (there was uncolored element in D_i). The condition clearly still holds.

When Bob plays with this strategy, the game could have stopped, because of few reasons. Alice does not have an admissible move, then she loses. For all $i \leq h$ there is equality $d_i = k$, but then by looking at $M_k|_{C \cup D_i}$ it has rank $2k$, so $2c_i \leq c_i + d_i \leq 2k$, so there can be at most k elements of color i in C , but $hk < (2k - 1)k = |C|$, so Alice lost. And possibly the game has ended because the whole matroid has been colored, but then also $c_i \leq d_i = k$, so this is not the case.

It remains to justify that coloring the elements of $D_1 \cup \dots \cup D_{3k(2k-1)}$ by Alice can not help her. To see this we modify the invariant that Bob wants to keep. For any $i \leq h$, let ϵ_i be the number of sets among D_i, D_{i+h} with some elements colored by i . The condition says that always after his move, number d_i of elements colored by i in $D_i \cup D_{i+h}$ is greater or equal to $c_i + \epsilon_i - 1$, where c_i is the number of elements in C colored by i . It can be shown analogously to the previous case that this invariant can be kept, under the condition that Bob has "enough space". This condition also gives the same consequences. Let \mathcal{D}_i be the union of those of D_i, D_{i+h} , which have an element colored by i . We see that $M_k|_{C \cup \mathcal{D}_i}$ has rank $2k - 1 + \epsilon_i$, so $c_i + c_i + \epsilon_i - 1 \leq c_i + d_i \leq 2k - 1 + \epsilon_i$. Therefore there can be at most k elements in C colored by i .

Additionally to provide "enough space" we add a condition that whenever Alice colors an element of $D_i \cup D_{i+h}$, then Bob colors another element of this set with i .

To avoid one very special case we have to add one more extra condition. Namely, if Alice colors an element of some D_a , $a \in \{i, i + h, i + 2h\} = \{a, b, c\}$ for the first time (Bob has not colored anything yet with i), then he colors with i an element of D_b . Later he will color with i only elements of D_b and D_c , and treat them as D_i, D_{i+h} . This completes the proof. \square

REMARK 1. *Suppose that Alice has won in the game on M_k , with $2k - 1$ colors, while Bob was using the above strategy, $k > 2$. Then exactly k elements of part C will be colored with i , for $i = 1, \dots, 2k - 1$. Additionally the inequalities showing a contradiction in the proof of Theorem 4 become equalities. So M_k are extremal examples for which $2k - 1$ colors suffice for Alice.*

REMARK 2. *Matroid M_k satisfies $\chi_f(M_k) = k$. By looking carefully at the argument from the proof of Theorem 4 we get that $\chi_{f,g}(M_k) \geq 2k - 1$ for $k > 2$.*

Our results lead to the following unsolved case:

QUESTION 4. *Is the bound $\chi_g(M) \leq 2\chi(M) - 1$ true for arbitrary matroids?*

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